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The High Temperature Region of the Viana–Bray Diluted Spin Glass Model*

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In this paper, we study the high temperature or low connectivity phase of the Viana–Bray model in the absence of magnetic field. This is a diluted version of the well known Sherrington–Kirkpatrick mean field spin glass. In the whole replica symmetric region, we obtain a complete control of the system, proving annealing for the infinite volume free energy and a central limit theorem for the suitably rescaled fluctuations of the multi-overlaps. Moreover, we show that free energy fluctuations, on the scale $1/N$, converge in the infinite volume limit to a non-Gaussian random variable, whose variance diverges at the boundary of the replica-symmetric region. The connection with the fully connected Sherrington–Kirkpatrick model is discussed.

KEY WORDS: Disordered systems; diluted spinglasses; Viana-Bray model; replica symmetry; multi-overlaps.

1. INTRODUCTION

Diluted mean field spin glasses attract a great interest among physicists and probabilists, for at least two reasons. First of all, due to their finite degree of connectivity, they represent a sort of intermediate situation between fully connected models and realistic spin glasses with finite range interactions. Secondly, many random optimization problems arising in theoretical computer science are mapped in a natural way into the study of the ground

* Dedicated to Giovanni Jona Lasinio, in occasion of his 70th birthday.

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state of diluted mean field spin glass models. The mean field character of these systems makes them exactly solvable, at least in the framework of Parisi theory of replica symmetry breaking.[1] Recently, many results have been obtained in this direction, culminating in the resolution of the K-sat model within the framework of "one-step replica symmetry breaking" in ref. 2. Much less is know from the rigorous point of view, two remarkable exceptions being refs. 3 and 4. In ref. 3, through a suitable extension of the interpolation methods introduced in refs. 5 and 6 for fully connected models, S. Franz and M. Leone proved, for a wide class of diluted models, that the thermodynamic limit for the free energy density exists, and that it is bounded below by Parisi solution with replica symmetry breaking. In ref. 4, instead, M. Talagrand proved that replica symmetry holds for sufficiently high temperature or low average connectivity.

In the present work we concentrate on the case of the Viana–Bray model,[7,8] where each spin interacts through two body couplings of random sign with a *finite* random number of other spins, even in the infinite volume limit. This is a diluted version of the well known Sherrington–Kirkpatrick (SK) model.[1,9] We identify the replica symmetric region, and we obtain a complete control of the system there. In particular, through a suitable extension of the "quadratic replica coupling method" we introduced in ref. 10, we prove that annealing holds for the free energy, in the infinite volume limit. Moreover, as in ref. 11, we prove limit theorems for the fluctuations of (multi)-overlaps and of the free energy. While the fluctuations of the multi-overlaps on the scale $1/\sqrt{N}$ turn out to be Gaussian in the infinite volume limit, like for the SK model, free energy fluctuations (on the scale $1/N$) tend to a non-Gaussian random variable, whose variance diverges at the boundary of the replica symmetric region. The validity of our method depends crucially on the assumption that there is no magnetic field. Indeed, it is only in this situation that, at high temperature and low connectivity, annealing holds for the free energy and the order parameter is trivial, all multi-overlaps being essentially zero.

The organization of the paper is as follows. In Section 2 we give the basic definitions concerning the model, and in Section 3 we discuss the role played by the multi-overlaps in its thermodynamical description. The relationship between the model under consideration and the fully connected one is considered in Section 4. In Sections 5 and 6, we identify the replica symmetric region and we prove annealing for the free energy. Finally, in Sections 7 and 8 we prove limit theorems for the fluctuations in the annealed region, while Section 9 is dedicated to conclusions and outlook to future developments.

2. DEFINITION OF THE MODEL

The Hamiltonian of the Viana–Bray model,[7] for a given configuration of the N Ising spin variables $\sigma_i = \pm 1$, $i = 1,\dots, N$, is defined as

$$H_N(\sigma, \alpha; \mathcal{J}) = -\sum_{\mu=1}^{\xi_{\alpha N}} J_\mu \sigma_{i_\mu} \sigma_{j_\mu}. \tag{1}$$

Here, $\xi_{\alpha N}$ is a Poisson random variable of mean value αN, for some $\alpha > 0$, i.e.,

$$P(\xi_{\alpha N} = k) = \pi(k, \alpha N) \equiv e^{-\alpha N} \frac{(\alpha N)^k}{k!} \qquad k = 0, 1, 2,\dots, \tag{2}$$

while $\{J_\mu\}$ is a family of independent identically distributed (i.i.d.) symmetric random variables, and the integer valued random variables i_μ, j_μ are independent of each other, as well as of $\xi_{\alpha N}$ and of the J_μ's, and are uniformly distributed on the set $\{1, 2,\dots, N\}$. We denote by $\mathcal{J}$ the dependence of the Hamiltonian on the whole set of quenched disordered variables $\xi_{\alpha N}$, J_μ, i_μ, j_μ. The parameter α fixes the average degree of connectivity of the system. Indeed the number of different sites, which interact with a given spin variable, behaves approximately like a Poisson random variable of parameter 2α, for large values of N. This is to be compared with the case of the SK model, where any spin interacts with all the other $N-1$. A second important difference with respect to the SK model is that, in the present case, the infinite volume limit of the system *does* depend on the probability distribution $\rho(J)$ of J_μ. In the case $\rho(J) = 1/2(\delta(J-1) + \delta(J+1))$, the Viana–Bray model is closely related to the so called 2-XOR-SAT problem[12] of computer science. In the course of this work, we do not specify the form of $\rho(J)$, but for simplicity we assume J to be a bounded random variable

$$|J| \leqslant 1. \tag{3}$$

More general cases can be considered, at the expense of some additional technical work.

The partition function $Z_N(\beta, \alpha; \mathcal{J})$, the disorder dependent free energy $f_N(\beta, \alpha; \mathcal{J})$, the Gibbs state $\omega_{\mathcal{J}}$ and the quenched free energy $-\beta A_N(\beta, \alpha)$ are defined in the usual way, for a given value of the inverse temperature β:

$$Z_N(\beta, \alpha; \mathscr{J}) = \sum_{\{\sigma\}} e^{-\beta H_N(\sigma, \alpha; \mathscr{J})} \tag{4}$$

$$f_N(\beta, \alpha; \mathscr{J}) = -\frac{1}{N\beta} \ln Z_N(\beta, \alpha; \mathscr{J}) \tag{5}$$

$$\omega_{\mathscr{J}}(\mathscr{O}) = Z_N(\beta, \alpha; \mathscr{J})^{-1} \sum_{\{\sigma\}} \mathscr{O}(\sigma)\, e^{-\beta H_N(\sigma, \alpha; \mathscr{J})} \tag{6}$$

$$A_N(\beta, \alpha) = \frac{1}{N} E \ln Z_N(\beta, \alpha; \mathscr{J}) = -\beta E f_N(\beta, \alpha; \mathscr{J}). \tag{7}$$

Here, $\mathscr{O}$ is a generic function of the spin variables, and E denotes expectation with respect to all quenched random variables:

$$E(.) = E_{\xi_{\alpha N}} E_{\{J_\mu\}} E_{\{i_\mu\}} E_{\{j_\mu\}}(.). \tag{8}$$

Like in the case of fully connected models, it is possible to prove that $f_N(\beta, \alpha; \mathscr{J})$ is self-averaging when the system size grows to infinity, and to give bounds, exponentially small in N, on the probability of its fluctuations. The precise result is stated and proved in Appendix A.

As usual, one introduces *real* replicas as independent identical copies of the system, subject to the same disorder realization, and denotes with $\Omega_{\mathscr{J}}(.)$ the disorder dependent product Gibbs state

$$\Omega_{\mathscr{J}} = \omega_{\mathscr{J}}^{(1)} \otimes \omega_{\mathscr{J}}^{(2)} \otimes \cdots, \tag{9}$$

where the state $\omega_{\mathscr{J}}^{(a)}$ acts on the ath replica. Moreover, the average $\langle . \rangle$, involving both thermal and disorder averages, is defined as

$$\langle . \rangle = E\Omega_{\mathscr{J}}(.). \tag{10}$$

A very important role is played by the multi-overlaps between n configurations $\sigma^{(1)},\ldots, \sigma^{(n)}$, defined as

$$q_{1\cdots n} = \frac{1}{N} \sum_{i=1}^{N} \sigma_i^{(1)} \cdots \sigma_i^{(n)}. \tag{11}$$

Of course,

$$-1 \leqslant q_{1\cdots n} \leqslant 1. \tag{12}$$

Notice that for $n=2$ one recovers the usual definition of the overlap as normalized scalar product between two configurations.

While for fully connected models the whole physical content of the theory is encoded in the probability distribution of the overlaps,[1] all multi-overlaps play an essential role in the present case.[7,8] In Section 4 we will show how, when the limit of infinite connectivity is suitably performed, the multi-overlaps with $n > 2$ become inessential.

3. THE ROLE OF THE MULTI-OVERLAPS

An important ingredient of the methods employed in ref. 3 is a smart use of the properties of the Poisson random variables. Indeed, while the choice of the Poisson distribution for the number $\xi_{\alpha N}$ of terms appearing in the Hamiltonian (1) is in principle not essential (any other random variable sharply concentrated around the value αN would yield an equivalent model, in the infinite volume limit), it turns out to be a great technical simplification. The basic elementary properties one employs, for the distribution function of a Poisson random variable ξ_λ of parameter $\lambda > 0$, are

$$k\pi(k, \lambda) = \lambda\pi(k-1, \lambda) \tag{13}$$

and

$$\frac{d}{d\lambda}\pi(k, \lambda) = -\pi(k, \lambda) + \pi(k-1, \lambda)(1-\delta_{k,0}). \tag{14}$$

In a sense, Eq. (13) replaces the identity

$$EJF(J) = EF'(J), \tag{15}$$

which plays a fundamental role in the study of the fully connected models, and which holds for any smooth function F if J is a Gaussian standard random variable.

For instance, let us show how Eq. (13) allows to express the internal energy of the Viana–Bray model as a sum of simple averages involving multi-overlaps. For an analogous computation, see ref. 3. One has

$$-\frac{\partial}{\partial\beta}A_N(\beta, \alpha) = \frac{\langle H\rangle}{N} = -\frac{1}{N}\sum_{k=1}^{\infty}\pi(k, \alpha N)\sum_{\mu=1}^{k}\langle J_\mu\sigma_{i_\mu}\sigma_{j_\mu}\rangle_k, \tag{16}$$

where $\langle\,.\,\rangle_k$ denotes the average where the value of the random variable $\xi_{\alpha N}$ has been fixed to k. Then, using property (13),

$$\frac{\langle H\rangle}{N} = -\frac{1}{N}\sum_{k=1}^{\infty}k\pi(k, \alpha N)\langle J_k\sigma_{i_k}\sigma_{j_k}\rangle_k = -\alpha\sum_{k=1}^{\infty}\pi(k-1, \alpha N)\langle J_k\sigma_{i_k}\sigma_{j_k}\rangle_k. \tag{17}$$

Now, we use the identity

$$\langle J_k \sigma_{i_k} \sigma_{j_k} \rangle_k = E\omega_{\mathscr{J}}(J_k \sigma_{i_k} \sigma_{j_k})_k = E\,\frac{\omega_{\mathscr{J}}(J_k \sigma_{i_k} \sigma_{j_k} \exp(\beta J_k \sigma_{i_k} \sigma_{j_k}))_{k-1}}{\omega_{\mathscr{J}}(\exp(\beta J_k \sigma_{i_k} \sigma_{j_k}))_{k-1}}, \tag{18}$$

to rewrite (17) as

$$\frac{\langle H \rangle}{N} = -\alpha \sum_{k=0}^{\infty} \pi(k, \alpha N)\, E\,\frac{\omega_{\mathscr{J}}(J\sigma_i \sigma_j \exp(\beta J \sigma_i \sigma_j))_k}{\omega_{\mathscr{J}}(\exp(\beta J \sigma_i \sigma_j))_k}, \tag{19}$$

where the random variables J, i, j denote independent copies of J_μ, i_μ, j_μ, respectively. Finally, recalling that i and j are uniformly distributed over $\{1,..., N\}$, one finds

$$\frac{\langle H \rangle}{N} = -\alpha E\,\frac{\sum_{i,j=1}^{N}}{N^2}\,\frac{\omega_{\mathscr{J}}(J\sigma_i \sigma_j \exp(\beta J \sigma_i \sigma_j))}{\omega_{\mathscr{J}}(\exp(\beta J \sigma_i \sigma_j))} \tag{20}$$

$$= -\frac{\alpha}{N^2} \sum_{i,j=1}^{N} EJ\,\frac{\tanh(\beta J) + \omega_{\mathscr{J}}(\sigma_i \sigma_j)}{1 + \tanh(\beta J)\, \omega_{\mathscr{J}}(\sigma_i \sigma_j)}. \tag{21}$$

Notice that we have employed the identity

$$e^{\beta J \sigma_i \sigma_j} = \cosh(\beta J) + \sigma_i \sigma_j \sinh(\beta J) \tag{22}$$

in the last step. Thanks to (3), $|\tanh(\beta J)| \leqslant \tanh \beta < 1$ so that the expression in (21) can be expanded in absolutely convergent Taylor series around $\tanh(\beta J) = 0$. Recalling the definition of the multi-overlaps and the symmetry of the random variable J, one finally finds

$$\begin{aligned} -\frac{\partial}{\partial \beta} A_N(\beta, \alpha) = {} & \frac{\langle H \rangle}{N} \\ = {} & -\alpha E(J \tanh(\beta J)) \\ & + \alpha \sum_{n=0}^{\infty} \langle q_{1\cdots 2n+2}^2 \rangle\, E\{J \tanh^{2n+1}(\beta J)(1 - \tanh^2(\beta J))\}. \end{aligned} \tag{23}$$

Indeed, it follows from the definitions (9), (10) of the averages $\Omega_{\mathscr{J}}(.)$ and $\langle . \rangle$ that

$$\frac{1}{N^2} \sum_{i,j=1}^{N} E\omega_{\mathscr{J}}^{2n}(\sigma_i \sigma_j) = \frac{1}{N^2} \sum_{i,j=1}^{N} \Omega_{\mathscr{J}}(\sigma_i^{(1)} \cdots \sigma_i^{(2n)} \sigma_j^{(1)} \cdots \sigma_j^{(2n)}) = \langle q_{1\cdots 2n}^2 \rangle. \tag{24}$$

In the particular case where $J_\mu = \pm 1$, the above expression (23) reduces to

$$-\frac{\partial}{\partial \beta} A_N(\beta, \alpha) = -\alpha \tanh \beta + \alpha \sum_{n=0}^{\infty} (\tanh \beta)^{2n+1} (1 - \tanh^2 \beta) \langle q^2_{1\cdots 2n+2} \rangle. \tag{25}$$

4. THE INFINITE CONNECTIVITY LIMIT AND THE SK MODEL

In this section, we discuss the relationship between the Viana–Bray and the fully connected SK model. As it was already observed in refs. 7 and 8, the latter is obtained when the average connectivity α tends to infinity, provided that the strength of the couplings J_μ, or equivalently the inverse temperature, is suitably rescaled to zero. Let us discuss this point in greater detail. To this purpose, recall that the SK model in zero external field is defined by the Hamiltonian

$$H_N^{\text{S.K.}}(\sigma; J) = -\frac{1}{\sqrt{N}} \sum_{1 \leqslant i < j \leqslant N} J_{ij} \sigma_i \sigma_j, \tag{26}$$

where the couplings J_{ij} are i.i.d. centered Gaussian random variables of unit variance. Now, we want to compare the Viana–Bray model, with parameters β and α, with the SK model, at an inverse temperature β' defined as

$$\beta'^2 = 2\alpha E \tanh^2(\beta J). \tag{27}$$

In particular we show that, in the limit $\alpha \to \infty$, $\beta \to 0$ with $\beta' = \text{const}$, one has

$$\left\{ \lim_{N \to \infty} \frac{1}{N} E \ln Z_N(\beta, \alpha; \mathscr{J}) \right\} \xrightarrow{\alpha \to \infty} \left\{ \lim_{N \to \infty} \frac{1}{N} E \ln Z_N^{\text{S.K.}}(\beta'; J) \right\}. \tag{28}$$

To this purpose, let $0 \leqslant t \leqslant 1$ and define an auxiliary partition function $Z_N(t)$ as

$$Z_N(t) = \sum_{\{\sigma\}} \exp\left(\beta \sum_{\mu=1}^{\xi_{\alpha N t}} J_\mu \sigma_{i_\mu} \sigma_{j_\mu} + \beta' \sqrt{\frac{1-t}{N}} \sum_{1 \leqslant i < j \leqslant N} J_{ij} \sigma_i \sigma_j \right). \tag{29}$$

Of course, for $t = 1$ one recovers the partition function (4) of the diluted model, while for $t = 0$ one has the partition function of the fully connected model, at inverse temperature β'. The t derivative of $1/NE \ln Z_N(t)$ can be performed along the lines of the computation of $\partial_\beta A_N(\beta, \alpha)$ in the previous section, with the result

$$\frac{d}{dt}\frac{1}{N} E \ln Z_N(t) = \alpha\left(E \ln \cosh(\beta J) - \frac{1}{2}\sum_{n=1}^{\infty} \frac{E \tanh^{2n}(\beta J)}{n} \langle q^2_{1\cdots 2n}\rangle\right)$$

$$-\frac{\beta'^2}{4}(1-\langle q^2_{12}\rangle) \tag{30}$$

$$=\left(\alpha E \ln \cosh(\beta J) - \frac{\beta'^2}{4}\right) - \frac{\alpha}{2}\sum_{n=2}^{\infty} \frac{E(\tanh^{2n}(\beta J))}{n} \langle q^2_{1\cdots 2n}\rangle. \tag{31}$$

The term (30) derives from the t dependence of the Poisson random variable $\xi_{\alpha N t}$ in (29), while (31) comes from the $\sqrt{1-t}$ factor which multiplies the SK Hamiltonian. Before we proceed, let us notice that we have proved the inequality, uniform in N,

$$\frac{d}{dt}\frac{1}{N} E \ln Z_N(t) \leqslant \alpha E \ln \cosh(\beta J) - \frac{\beta'^2}{4}, \tag{32}$$

whose implications will be discussed below. Now, it is easy to see that the t derivative we are considering in (30) vanishes uniformly in N for $\beta \to 0$, $\alpha \to \infty$, if the constraint (27) is satisfied. Indeed, for $\alpha \to \infty$ Eq. (27) reduces to

$$2\alpha\beta^2 E J^2 = \beta'^2 + O\left(\frac{1}{\alpha}\right), \tag{33}$$

so that

$$\alpha E \ln \cosh(\beta J) - \frac{\beta'^2}{4} = \alpha E \ln\left(1+\frac{\beta^2 J^2}{2}\right) - \frac{\beta'^2}{4} + O\left(\frac{1}{\alpha}\right) = O\left(\frac{1}{\alpha}\right) \tag{34}$$

and

$$\frac{\alpha}{2}\sum_{n=2}^{\infty} \frac{E \tanh^{2n}(\beta J)}{n} \langle q^2_{1\cdots 2n}\rangle \leqslant \frac{\alpha}{2}\sum_{n=2}^{\infty} \frac{\beta^{2n}}{n} \xrightarrow{\alpha\to\infty} 0, \tag{35}$$

which concludes the proof of (28). ∎

5. THE REPLICA SYMMETRIC BOUND AND THE ANNEALED REGION

In ref. 6 it was proven that the Parisi solution for the SK model, with an arbitrary number of levels of replica symmetry breaking, is a lower bound for the free energy, at any temperature. Along the same lines, this result was extended in ref. 3 to the case of diluted models. In this context, one has to face the additional difficulty that, even at the level of the replica symmetric approximation, the Parisi order parameter is a function[8] (the probability distribution of the effective field) rather than a single number, as it happens instead for fully connected models.[1] In the present section, we recall briefly the replica symmetric bound for the Viana–Bray model under consideration, and we discuss the high temperature or low connectivity phase, where this bound actually gives the correct limit.

Let g be an arbitrary symmetric random variable (we assume its distribution to be regular enough to guarantee that all expressions below are well defined), and define the random variable u as

$$\tanh(\beta u) = \tanh(\beta J)\tanh(\beta g). \tag{36}$$

Here, J is distributed like any of the couplings J_μ and is independent of them (as well as of g). For given β and α, the replica symmetric trial functional $F_{\rm RS}(\beta, \alpha; g)$ is defined as

$$\begin{aligned} F_{\rm RS}(\beta,\alpha;g) = \ln 2 + \alpha E \ln\cosh(\beta J) + E\ln\cosh\left(\beta\sum_{\ell=1}^{\xi_{2\alpha}} u_\ell\right) \\ -2\alpha E\ln\cosh(\beta u) \\ -\frac{\alpha 2}{E}\ln(1-\tanh^2(\beta J)\tanh^2(\beta g_1)\tanh^2(\beta g_2)). \end{aligned} \tag{37}$$

Here, u_ℓ are independent copies of u and g_1, g_2 are independent copies of g. Then, one has[3]

$$\frac{1}{N} E\ln Z_N(\beta,\alpha;\mathcal{J}) \leqslant \inf_g F_{\rm RS}(\beta,\alpha;g) + O\left(\frac{1}{N}\right), \tag{38}$$

where the infimum is taken over the space of symmetric random variables g. It is not difficult to see, computing the functional derivative of $F_{\rm RS}(\beta, \alpha; g)$ with respect to the probability distribution $P(g)$ of g, that a sufficient condition of extremality for the replica symmetric functional is[8]

$$g \stackrel{d}{=} \sum_{\ell=1}^{\xi_{2\alpha}} u_\ell = \frac{1}{\beta}\sum_{\ell=1}^{\xi_{2\alpha}} \tanh^{-1}(\tanh(\beta J_\ell)\tanh(\beta g_\ell)), \tag{39}$$

where the equality holds in distribution. It is clear that the above equation always admits the trivial solution g concentrated at the value zero, i.e., with $P(g)=\delta(g)$. In this case, it follows from Eqs. (36), (37) that

$$F_{\rm RS}(\beta, \alpha; g\equiv 0) = \ln 2 + \alpha E \ln \cosh(\beta J) \tag{40}$$

which corresponds to take the expectation with respect to the random coupling signs before the logarithm, in the definition (7) of $A_N(\beta, \alpha)$:

$$F_{\rm RS}(\beta, \alpha; 0) = \frac{1}{N} E \ln E_{\{\mathrm{sign}(J_\mu)\}} Z_N(\beta, \alpha; \mathscr{J}). \tag{41}$$

In the following, we call $-1/\beta F_{\rm RS}(\beta, \alpha; 0)$ the "annealed free energy," even if strictly speaking in (41) we are performing an annealed average only on the signs of the couplings and not on their absolute values. The following result shows that, in a certain region of the parameters β and α, the trivial solution of (39) is actually the only one:

Proposition 1. If

$$2\alpha E \tanh^2(\beta J) < 1 \qquad \text{(annealed region)}, \tag{42}$$

the only symmetric random variable g satisfying Eq. (39) is the degenerate one: $P(g)=\delta(g)$.

Notice that, for $\alpha < 1/2$, the annealed region extends up to $\beta=\infty$.

Proof of Proposition 1. Let

$$\phi(v) = E e^{ivg} \tag{43}$$

be the characteristic function of g, which can be rewritten, thanks to condition (39) and to the Poisson distribution of $\xi_{2\alpha}$, as

$$\ln \phi(v) = 2\alpha \left(E \exp\left(i \frac{v}{\beta} \tanh^{-1}(\tanh(\beta J) \tanh(\beta g)) \right) - 1 \right). \tag{44}$$

This implies that

$$|\ln \phi(v)| \leqslant 2\alpha\, |v| \sqrt{2\alpha E \tanh^2(\beta J)}, \tag{45}$$

where we used the fact that

$$E g^2 \leqslant 2\alpha,$$

as it easily follows from (39) and from $|J| \leqslant 1$. Now, one can iterate the procedure, replacing the random variable g which appears at the right hand side of (44) with the expression given by Eq. (39), and so on. At the nth step of the iteration one has the bound

$$|\ln \phi(v)| \leqslant 2\alpha\, |v|\, (2\alpha E \tanh^2(\beta J))^{n/2}. \tag{46}$$

which goes to zero when $n \to \infty$, if condition (42) holds. ∎

On the other hand it is easy to realize that, outside the annealed region, the choice of the identically vanishing g does not realize the infimum in (38). Indeed, consider even the simple case of a two-valued random variable g with distribution

$$P(g) = \tfrac{1}{2}\,(\delta(g - g_0) + \delta(g + g_0)).$$

When $g_0 \simeq 0$, one finds

$$F_{\mathrm{RS}}(\beta, \alpha; g) - F_{\mathrm{RS}}(\beta, \alpha; g \equiv 0) = \frac{\alpha}{2}\, \beta^4 g_0^4 (1 - 2\alpha E \tanh^2(\beta J)) + O(g_0^6), \tag{47}$$

which is negative if (42) does not hold.

It is interesting to observe that breaking of annealing outside the region (42) can also be proved through a comparison with the SK model. Indeed, integration of the inequality (32) with respect to t between 0 and 1 gives

$$\ln 2 + \alpha E \ln \cosh(\beta J) - \frac{1}{N} E \ln Z_N(\beta, \alpha; \mathscr{J})$$

$$\geqslant \ln 2 + \frac{\beta'^2}{4} - \frac{1}{N} E \ln Z_N^{\mathrm{S.K.}}(\beta'; J),$$

i.e., the difference between the quenched and the annealed free energies is larger (in absolute value) for the diluted model than for its fully connected counterpart if β, α, and β' are related by the condition (27). Therefore, since it is well known that

$$\lim_{N \to \infty} \frac{1}{N} E \ln Z_N^{\mathrm{S.K.}}(\beta'; J) < \ln 2 + \frac{\beta'^2}{4} \tag{48}$$

for $\beta' > 1$, one has immediately breakdown of annealing for the Viana–Bray model, when $\beta'^2 = 2\alpha E \tanh^2(\beta J) > 1$.

6. CONTROL OF THE ANNEALED REGION

In the present section we prove that annealing actually holds for the Viana–Bray model in the region of parameters (42), i.e., that

Theorem 1. For $2\alpha E \tanh^2(\beta J) < 1$,

$$\frac{1}{N} E \ln Z_N(\beta, \alpha; \mathscr{J}) = \ln 2 + \alpha E \ln \cosh(\beta J) + O\left(\frac{1}{N}\right). \tag{49}$$

A related result was proven in ref. 13 by means of a cluster (or low connectivity) expansion.

We prove the theorem via a suitable adaptation of the "quadratic replica coupling" method we introduced in ref. 10 for the SK model. While the above result can also be obtained through the "second moment method,"[14] which consists in showing that

$$\frac{1}{N} \ln E(Z_N)^2 = \frac{1}{N} \ln(EZ_N)^2 + o(1), \tag{50}$$

the quadratic method we employ allows us to obtain self-averaging of the multi-overlaps in a stronger form, and to prove limit theorems for the fluctuations, as shown in the next two sections.

Consider a system of two coupled replicas of the model, defined by the partition function

$$Z_N^{(2)}(\beta, \alpha, \lambda; \mathscr{J}) = \sum_{\{\sigma^1, \sigma^2\}} e^{-\beta H_N(\sigma^1, \alpha; \mathscr{J}) - \beta H_N(\sigma^2, \alpha; \mathscr{J}) + N \frac{\lambda}{2} q_{12}^2}, \tag{51}$$

where $\lambda \geqslant 0$. Notice that the quadratic interaction gives a large weight to the pairs of configurations whose overlap is different from zero. Like in ref. 10 the idea is to show that, if λ is not too large, the interaction does not modify the infinite volume free energy density, so that q_{12} must be typically close to zero. Indeed, we can prove

Theorem 2. In the region

$$(\lambda + 2\alpha E \tanh^2(\beta J)) < 1, \qquad \lambda, \alpha \geqslant 0, \tag{52}$$

one has

$$\frac{1}{2N} E \ln Z_N^{(2)}(\beta, \alpha, \lambda; \mathscr{J}) = \ln 2 + \alpha E \ln \cosh(\beta J) + O\left(\frac{1}{N}\right) \tag{53}$$

and

$$\langle q_{1\cdots 2n}^2 \rangle \leqslant \langle q_{12}^2 \rangle = O\left(\frac{1}{N}\right). \tag{54}$$

Of course, this result implies the previous Theorem 2 since, for $\lambda = 0$,

$$\frac{1}{2N} E \ln Z_N^{(2)}(\beta, \alpha, 0; \mathscr{J}) = \frac{1}{N} E \ln Z_N(\beta, \alpha; \mathscr{J}).$$

Proof of Theorem 2. First of all, since[3]

$$\frac{\partial}{\partial \alpha} \frac{1}{N} E \ln Z_N(\beta, \alpha; \mathscr{J}) = E \ln \cosh(\beta J) - \frac{1}{2} \sum_{n=1}^{\infty} \frac{E \tanh^{2n}(\beta J)}{n} \langle q_{1\cdots 2n}^2 \rangle, \tag{55}$$

and

$$\langle q_{1\cdots 2n}^2 \rangle = \frac{1}{N^2} \sum_{i,j=1}^{N} E\omega_{\mathscr{J}}^{2n}(\sigma_i \sigma_j) \leqslant \frac{1}{N^2} \sum_{i,j=1}^{N} E\omega_{\mathscr{J}}^{2}(\sigma_i \sigma_j) = \langle q_{12}^2 \rangle, \tag{56}$$

one can write

$$\frac{\partial}{\partial \alpha}\left(F_{\mathrm{RS}}(\beta, \alpha; 0) - \frac{1}{N} E \ln Z_N\right) \leqslant \frac{\langle q_{12}^2 \rangle}{2} E \ln(1 - \tanh^2(\beta J))^{-1}. \tag{57}$$

Therefore, using convexity of $\ln Z_N^{(2)}$ with respect to λ and the identity

$$\frac{\partial}{\partial \lambda} \frac{1}{2N} E \ln Z_N^{(2)}(\beta, \alpha, \lambda; \mathscr{J})\Big|_{\lambda=0} = \frac{1}{4} \langle q_{12}^2 \rangle, \tag{58}$$

which follows from definition (51), one has

$$\frac{\partial}{\partial \alpha}\left(F_{\mathrm{RS}}(\beta, \alpha; 0) - \frac{E \ln Z_N}{N}\right)$$
$$\leqslant \frac{2E \ln(1 - \tanh^2(\beta J))^{-1}}{\lambda} \left(\frac{E \ln Z_N^{(2)}(\lambda)}{2N} - \frac{E \ln Z_N}{N}\right). \tag{59}$$

Next, we need an upper bound for $1/(2N)\, E \ln Z_N^{(2)}$ in terms of $F_{\mathrm{RS}}(\beta, \alpha; 0)$. To this purpose, we take λ to depend on α as $\lambda(\alpha) = \lambda_0 - 2\alpha E \tanh^2(\beta J)$, and we compute

$$\frac{d}{d\alpha}\frac{1}{2N}E\ln Z_N^{(2)}(\beta,\alpha,\lambda(\alpha);\mathcal{J})$$
$$=-\frac{1}{2}E\tanh^2(\beta J)\langle q_{12}^2\rangle_{\alpha,\lambda(\alpha)}+E\ln\cosh(\beta J)$$
$$+\frac{1}{4N^2}\sum_{i,j=1}^{N}E\ln[(1+\tanh^2(\beta J)\,\Omega_{\alpha,\lambda(\alpha)}(\sigma_i^1\sigma_j^1\sigma_i^2\sigma_j^2))^2$$
$$-4\tanh^2(\beta J)\,\omega^2_{\alpha,\lambda(\alpha)}(\sigma_i\sigma_j)]. \tag{60}$$

The computation is very similar in spirit to that of $\partial_\beta A_N$ in Section 3, the essential ingredients to be employed being Eq. (14) and the symmetry of J. Here, the averages refer to the coupled system with parameters α, $\lambda(\alpha)$. Then,

$$\frac{d}{d\alpha}\frac{1}{2N}E\ln Z_N^{(2)}(\beta,\alpha,\lambda(\alpha);\mathcal{J})$$
$$\leqslant-\frac{1}{2}E\tanh^2(\beta J)\langle q_{12}^2\rangle_{\alpha,\lambda(\alpha)}+E\ln\cosh(\beta J)$$
$$+\frac{1}{2}\ln(1+E\tanh^2(\beta J)\langle q_{12}^2\rangle) \tag{61}$$
$$\leqslant E\ln\cosh(\beta J), \tag{62}$$

where we used Jensen's inequality to take expectation inside the logarithm, together with the elementary estimate

$$\ln(1+x)\leqslant x.$$

Therefore, integrating between 0 and α one has

$$\frac{1}{2N}E\ln Z_N^{(2)}(\beta,\alpha,\lambda;\mathcal{J})\leqslant\alpha E\ln\cosh(\beta J)+\frac{1}{2N}\ln\sum_{\{\sigma^1,\sigma^2\}}e^{N\lambda_0 q_{12}^2/2}, \tag{63}$$

since at $\alpha=0$ only the quadratic replica coupling survives in the Hamiltonian. At this point, the proof proceeds exactly like in ref. 10: one introduces an auxiliary Gaussian standard random variable z with probability distribution

$$d\mu(z)=e^{-z^2/2}\frac{dz}{\sqrt{2\pi}}$$

and performs a simple rescaling, in order to linearize the dependence on q_{12} of the exponent:

$$\frac{1}{2N}\ln\sum_{\{\sigma^1,\sigma^2\}} e^{N\lambda_0 q_{12}^2/2} = \frac{1}{2N}\ln\sum_{\{\sigma^1,\sigma^2\}}\int e^{\sqrt{\lambda_0 N}\, q_{12} z}\, d\mu(z) \tag{64}$$

$$= \ln 2 + \frac{1}{2N}\ln\int\sqrt{\frac{N}{2\pi}}\exp N\left(-\frac{y^2}{2}+\ln\cosh(y\sqrt{\lambda_0})\right). \tag{65}$$

For $\lambda_0 = \lambda + 2\alpha E\tanh^2(\beta J) < 1$, one can employ the inequality

$$2\ln\cosh x \leqslant x^2 \tag{66}$$

to deduce, from Eqs. (63) and (65),

$$\frac{1}{2N}E\ln Z_N^{(2)}(\beta,\alpha,\lambda;\mathscr{J}) \leqslant \frac{1}{N}\ln EZ_N(\beta,\alpha;\mathscr{J}) + \frac{1}{4N}\ln\frac{1}{1-\lambda_0}, \tag{67}$$

so that Eq. (59) reduces to

$$\frac{\partial}{\partial\alpha}\left(F_{\mathrm{RS}}(\beta,\alpha;0) - \frac{E\ln Z_N}{N}\right)$$
$$\leqslant \frac{2E\ln(1-\tanh^2(\beta J))^{-1}}{\lambda}\left(F_{\mathrm{RS}}(\beta,\alpha;0) - \frac{E\ln Z_N}{N}\right) + O(N^{-1}). \tag{68}$$

Since

$$\left.\left(F_{\mathrm{RS}}(\beta,\alpha;0) - \frac{E\ln Z_N}{N}\right)\right|_{\alpha=0} = 0,$$

Eq. (68) implies, through a straightforward application of Gronwall's lemma, that

$$\left(F_{\mathrm{RS}}(\beta,\alpha;0) - \frac{E\ln Z_N}{N}\right) = O(N^{-1}), \tag{69}$$

for $2\alpha E\tanh^2(\beta J) < 1$.

Statement (53) then follows if one notices that, thanks to (67), (69) and to monotonicity of the free energy with respect to λ,

$$F_{RS}(\beta, \alpha; 0) + O(N^{-1}) = \frac{1}{N} E \ln Z_N \leqslant \frac{1}{2N} E \ln Z_N^{(2)}(\lambda)$$
$$\leqslant F_{RS}(\beta, \alpha; 0) + O(N^{-1}) \qquad (70)$$

in the region (52). Finally, statement (54) follows from (53) and from convexity of $E \ln Z_N^{(2)}$ with respect to λ. ∎

7. MULTI-OVERLAP FLUCTUATIONS IN THE ANNEALED REGION

In the previous section, we proved that the multi-overlap among any $2n$ configurations $\sigma^{(a_1)},\ldots, \sigma^{(a_{2n})}$ is typically small, in the annealed region. To study the infinite volume behavior of the multi-overlap fluctuations, we define

$$\eta_N^{a_1\cdots a_{2n}} = \sqrt{N}\, q_{a_1\cdots a_{2n}} \equiv \frac{1}{\sqrt{N}} \sum_{i=1}^{N} \sigma_i^{(a_1)} \cdots \sigma_i^{(a_{2n})}.$$

(Due to symmetry under permutation of the indices a_i, we will always assume them to be ordered as $a_1 < a_2 < \cdots < a_{2n}$.) Then, like for the SK model at high temperature,[11, 15, 16] one can prove that the rescaled (multi)-overlaps behave like independent centered Gaussian variables, in the infinite volume limit. Indeed, we prove the following

Theorem 3. In the annealed region (42), the variables $\eta_N^{a_1\cdots a_{2n}}$ converge in distribution, as $N \to \infty$, to centered Gaussian variables $\eta^{a_1\cdots a_{2n}}$ with variance

$$\langle (\eta^{a_1\cdots a_{2n}})^2 \rangle = \frac{1}{1 - 2\alpha E \tanh^{2n}(\beta J)}. \qquad (71)$$

Remark 1. Notice that, when the boundary of the annealed region (42) is approached, only the variance of $\eta^{a_1 a_2}$ diverges.

Remark 2. With the same method we employ to prove Theorem 3, it is possible to prove also that the limit random variables are jointly Gaussian and mutually independent, i.e.,

$$\langle \eta^{a_1\cdots a_{2n}} \eta^{b_1\cdots b_{2n}} \rangle = 0 \qquad \text{if} \quad \exists i\colon a_i \neq b_i \qquad (72)$$

$$\langle \eta^{a_1\cdots a_{2n}} \eta^{a_1\cdots a_{2n'}} \rangle = 0 \qquad \text{if} \quad n \neq n'. \qquad (73)$$

Proof of Theorem 3. We will prove only that

$$\phi_N(u) \equiv \langle e^{iu\eta_N^{12}} \rangle \longrightarrow \exp\left(-\frac{u^2}{2(1-2\alpha E \tanh^2(\beta J))} \right). \tag{74}$$

The extension of (71) to $n > 1$ involves heavier notations but no additional difficulty.

The proof is based on the cavity method,[1] which in essence consists in analyzing what happens when one removes one of the spins, thereby transforming the original system into one of size $N-1$. For applications of the cavity method in the context of mathematical physics, we refer to refs. 11, 17, and 18, and ref. 4. As in ref. 11, the idea is to write down a linear differential equation for $\phi_N(u)$, in the thermodynamic limit. First of all, using symmetry among sites one can write

$$\partial_u \phi_N(u) = i \langle \eta_N^{12} e^{iu\eta_N^{12}} \rangle = i\sqrt{N} \langle \sigma_N^1 \sigma_N^2 e^{iu\eta_N^{12}} \rangle. \tag{75}$$

Notice that, thanks to Theorem 2 of the previous section,

$$|\partial_u \phi_N(u)| \leqslant \langle (\eta_N^{12})^2 \rangle^{\frac{1}{2}} \leqslant C, \tag{76}$$

uniformly in N. Then, defining

$$u' = u\sqrt{1-1/N},$$

one has

$$\partial_u \phi_N(u) = i\sqrt{N} \langle \sigma_N^1 \sigma_N^2 \exp(iu\sigma_N^1 \sigma_N^2/\sqrt{N} + iu'\eta_{N-1}^{12}) \rangle \tag{77}$$

$$= -u\phi_N(u) + i\sqrt{N} \langle \sigma_N^1 \sigma_N^2 e^{iu'\eta_{N-1}^{12}} \rangle + o(1) \tag{78}$$

where the term $o(1)$, vanishing for $N \to \infty$, arises from the expansion of $\exp(iu\sigma_N^1 \sigma_N^2/\sqrt{N})$ around $u = 0$ and from the replacement $u' \to u$. Now consider the set

$$\mathscr{A} = \{\mathscr{J}: \nexists \mu: i_\mu = j_\mu = N\}, \tag{79}$$

where i_μ, j_μ are the random site indices appearing in (1). Since the probability of $\mathscr{A}$ is very close to one,

$$P(\mathscr{A}) = 1 - O(1/N),$$

one can write

$$i\sqrt{N}\langle\sigma_N^1\sigma_N^2 e^{iu'\eta_{N-1}^{12}}\rangle = i\sqrt{N}\langle\sigma_N^1\sigma_N^2 e^{iu'\eta_{N-1}^{12}} 1_{\mathscr{A}}\rangle + o(1) \tag{80}$$

where $1_{\mathscr{A}}$ is the indicator function of the set $\mathscr{A}$. Next, we single out all terms $-J_\nu\sigma_{i_\nu}\sigma_N$ in the Hamiltonian (1) involving the Nth spin (the number of these terms is a Poisson variable $\xi_{2\alpha}$ of mean value 2α) and we rewrite (80) as

$$i\sqrt{N}\,E\,\frac{\Omega'(e^{iu'\eta_{N-1}^{12}} Av\,\sigma_N^1\sigma_N^2 \exp(\beta\sum_{\ell=1}^2 \sigma_N^\ell \sum_{\nu=1}^{\xi_{2\alpha}} J_\nu\sigma_{i_\nu}^\ell))}{\Omega'(Av \exp(\beta\sum_{\ell=1}^2 \sigma_N^\ell \sum_{\nu=1}^{\xi_{2\alpha}} J_\nu\sigma_{i_\nu}^\ell))} \equiv i\sqrt{N}\,E\,\frac{A}{B}, \tag{81}$$

where Av denotes average on the two-valued unbiased variables $\sigma_N^\ell = \pm 1$ and $\Omega'(.)$ is the Gibbs average for a system with $N-1$ spins and connectivity parameter $\alpha' = \alpha(1-1/(N-1))$.[3] Of course, since we are restricting to the set $\mathscr{A}$, the indices i_ν are i.i.d. random variables uniformly distributed on $\{1,..., N-1\}$. Now, we show that the denominator B in (81) can be replaced by the random variable

$$\tilde{B} = \prod_{\nu=1}^{\xi_{2\alpha}} \cosh^2(\beta J_\nu), \tag{82}$$

by neglecting an error term which vanishes in the thermodynamic limit. To this purpose we use the obvious identity

$$E\,\frac{A}{B} = 2E\,\frac{A}{\tilde{B}} - E\,\frac{AB}{\tilde{B}^2} + E\,\frac{A}{B}\left(\frac{\tilde{B}-B}{\tilde{B}}\right)^2, \tag{83}$$

as it was done in ref. 18. As we will show below, the last term in the r.h.s. vanishes for $N \to \infty$. The first term is easily computed. Indeed, recalling the mutual independence of the variables J_ν, i_ν and using the formula

$$Ea^{\xi_\lambda} = e^{-\lambda(1-a)},$$

which holds for $a \neq 0$ if ξ_λ is a Poisson random variable of mean λ, one finds

$$i\sqrt{N}\,E\,\frac{A}{\tilde{B}} = i\sqrt{N}\,E\Omega'\left\{e^{iu'\eta_{N-1}^{12}} \sinh\left(2\alpha E\tanh^2(\beta J)\,\frac{\eta_{N-1}^{12}}{\sqrt{N-1}}\right)\right\}.$$

[3] This is because the average number of terms appearing in the modified Hamiltonian of the $N-1$ spin system is $N\alpha - 2\alpha \equiv \alpha'(N-1)$.

Then, expanding the sinh(...) at first order around zero and recalling that

$$\sup_N \langle (\eta_N^{12})^2 \rangle < \infty,$$

one has

$$\begin{aligned} i\sqrt{N}\, E\,\frac{A}{\tilde{B}} &= 2i\alpha E \tanh^2(\beta J)\, E\Omega'\{e^{iu'\eta_{N-1}^{12}}\eta_{N-1}^{12}\} + o(1) \\ &= 2\alpha E \tanh^2(\beta J)\, \partial_u \phi_N(u) + o(1). \end{aligned} \tag{84}$$

As for the second term in (83), one finds again

$$i\sqrt{N}\, E\,\frac{AB}{\tilde{B}^2} = 2\alpha E \tanh^2(\beta J)\, \partial_u \phi_N(u) + o(1). \tag{85}$$

Finally, we show that the last term can be neglected. First of all, one has

$$B \geqslant 1,$$

as it follows from Jensen inequality, interchanging the thermal average Ω' and the exponential in the definition of B. Therefore,

$$\sqrt{N}\left| E\,\frac{A}{B}\left(\frac{\tilde{B}-B}{\tilde{B}}\right)^2\right| \leqslant \sqrt{N}\, E e^{2\beta\xi_{2\alpha}}\left(1-\frac{B}{\tilde{B}}\right)^2. \tag{86}$$

The computation of (86) proceeds in analogy with that of $EA/\tilde{B}$. In this case, however, one finds that the dominant term in the Taylor expansion is of order

$$\frac{1}{\sqrt{N}}\langle (\eta_N^{12})^2 \rangle = o(1). \tag{87}$$

Therefore, recalling Eqs. (83)–(85), together with Eq. (77), we find that $\phi_N(u)$ solves the linear differential equation

$$(1-2\alpha E \tanh^2(\beta J))\, \partial_u \phi_N(u) = -u\phi_N(u) + o(1) \tag{88}$$

which, together with the obvious initial condition

$$\phi_N(0) = 1, \tag{89}$$

implies the result (74). ∎

8. FREE ENERGY FLUCTUATIONS

Is easy to realize that the Viana–Bray model resembles locally a spin glass model on a tree, where the number of branches starting at each node is a Poisson random variable of parameter 2α and the couplings associated to the branches are i.i.d. random variables J_μ. The non-triviality of the Viana–Bray model arises from the presence of loops of length $O(\ln N)$ in the underlying graph. For the model on the tree and without magnetic field, the computation of the partition function for any disorder realization is elementary,

$$Z_N^{\text{tree}}(\beta, \alpha; \mathcal{J}) = 2^N \prod_{\mu=1}^{\xi_{\alpha N}} \cosh(\beta J_\mu), \tag{90}$$

so that

$$\frac{1}{N} E \ln Z_N^{\text{tree}}(\beta, \alpha; \mathcal{J}) = \ln 2 + \alpha E \ln \cosh(\beta J). \tag{91}$$

Theorem 1 shows that, in the annealed region, the Viana–Bray model behaves like its tree-like counterpart, as far as only the infinite volume limit of the free energy density is concerned. However, the difference between the two models becomes evident if one looks at the difference between the respective free energies, on the scale $1/N$. Indeed, the following result holds:

Theorem 4. Define the random variable

$$\hat{f}_N(\beta, \alpha; \mathcal{J}) \equiv \ln Z_N(\beta, \alpha; \mathcal{J}) - \left(N \ln 2 + \sum_{\mu=1}^{\xi_{\alpha N}} \ln \cosh(\beta J_\mu) \right), \tag{92}$$

where $J_1,\dots, J_{\xi_{\alpha N}}$ are the same couplings which appear in the Hamiltonian (1). In the annealed region (42) $\hat{f}_N(\beta, \alpha; \mathcal{J})$ converges in distribution, as $N \to \infty$, to a non-Gaussian random variable $\hat{f}$ with characteristic function

$$E \exp(is\hat{f}) = \exp\left\{ -\frac{1}{2} \sum_{n=1}^{\infty} is(is-1)\cdots(is-(2n-1)) \frac{\ln(1-2\alpha E \tanh^{2n}(\beta J))}{(2n)!} \right\}. \tag{93}$$

The variance of the limit random variable diverges when the boundary of the annealed region is approached.

Remark. It is not difficult to check that, when the infinite connectivity limit is performed as in Section 4, the limit random variable becomes Gaussian (the terms of order higher than s^2 disappear in the series) and one recovers the well known result of ref. 19 for the fluctuations of the SK free energy at zero external field and $\beta' < 1$.

Proof of Theorem 4. The idea of the proof is to write down a linear differential equation for the characteristic function

$$\phi_N(\alpha, s) = E \exp(is\hat{f}_N). \tag{94}$$

Of course, for $\alpha = 0$ both the Viana–Bray and the tree model consist in an empty graph, so that

$$\phi_N(0, s) = 1. \tag{95}$$

As for the α derivative, the computation can be performed along the lines of the computation of $\partial_\beta A_N(\beta, \alpha)$ in Section 3, with the result

$$\frac{\partial \phi_N(\alpha, s)}{\partial \alpha} = -N\phi_N(\alpha, s) + \frac{1}{N} \sum_{i,j=1}^{N} E\, e^{is\hat{f}_N}(1 + \tanh(\beta J)\, \omega_{\mathcal{J}}(\sigma_i \sigma_j))^{is}. \tag{96}$$

Since $|\tanh(\beta J)| < \tanh \beta < 1$, one can expand the r.h.s. in an absolutely convergent Taylor series, using the formula

$$(1+x)^a = 1 + \sum_{n=1}^{\infty} \frac{a(a-1)\cdots(a-(n-1))}{n!} x^n$$

and write

$$\frac{\partial \phi_N(\alpha, s)}{\partial \alpha} = \sum_{n=1}^{\infty} E \tanh^{2n}(\beta J) \frac{is(is-1)\cdots(is-(2n-1))}{(2n)!} E e^{is\hat{f}} \Omega_{\mathcal{J}}(N q^2_{1\cdots 2n}). \tag{97}$$

Notice that, thanks to Theorem 2,

$$\langle N q^2_{1\cdots 2n} \rangle \leqslant \langle N q^2_{12} \rangle \leqslant \sup_N \langle N q^2_{12} \rangle < \infty$$

and the derivative in (97) can be bounded uniformly in N. Next, we can replace $\Omega_{\mathcal{J}}(N q^2_{1\cdots 2n})$ with $\langle N q^2_{1\cdots 2n} \rangle$. Indeed, thanks to Theorem 3 of the previous section,

$$\begin{aligned} &\langle (\Omega_{\mathcal{J}}(N q^2_{1\cdots 2n}) - \langle N q^2_{1\cdots 2n} \rangle)^2 \rangle \\ &\quad = \langle (\eta_N^{1\cdots 2n})^2\, (\eta_N^{2n+1\ldots 4n})^2 \rangle - \langle (\eta_N^{1\cdots 2n})^2 \rangle^2 = o(1). \end{aligned} \tag{98}$$

Therefore, denoting by ϕ the infinite volume limit of ϕ_N, one has

$$\frac{\partial \phi(\alpha, s)}{\partial \alpha} = \sum_{n=1}^{\infty} \frac{is(is-1)\cdots(is-(2n-1))}{(2n)!} \frac{E \tanh^{2n}(\beta J)}{1-2\alpha E \tanh^{2n}(\beta J)} \phi(\alpha, s), \tag{99}$$

from which the statement of the theorem follows after integration with respect to α. ▮

9. OUTLOOK AND CONCLUSIONS

In this paper, we have provided a complete picture of the high temperature or low connectivity phase of the Viana–Bray model without magnetic field, where annealing holds. Breaking of annealing is forecasted by the divergence of fluctuations of the free energy density (on the scale $1/N$) and of the two-replica overlap (on the scale $1/\sqrt{N}$). On the other hand, the fluctuations of the multi-overlap among $2n \geqslant 4$ configurations show no singularity when the boundary of the annealed region is approached.

The high temperature phase of the diluted p-spin model with $p > 2$ can be studied with the same techniques, but in this case one does not control the whole expected annealed region. On the other hand, the methods we presented here do not extend to the study of the replica symmetric region of the K-sat model, or of the diluted mean field model in presence of a magnetic field. In this case annealing does not hold, even at high temperature, and the random variable g which realizes the infimum of the replica symmetric functional is not trivial, as it is well known (see, for instance, ref. 20). We plan to report on this subject in a future paper.

APPENDIX A. SELF-AVERAGING OF FREE ENERGY AND GROUND STATE ENERGY DENSITIES

In this section we prove an upper bound, exponentially small in N and independent of β, for the fluctuations of the disorder dependent free energy density of the Viana–Bray model. Independence of β implies that the bound holds also for the fluctuations of the ground state energy density. Similar results have been known for a long time in the case of fully connected mean field spin glass models (for instance, see ref. 14 and references therein) and for some random optimization problems.[21, 22]

Theorem 5. For any value of β, α and N, one has

$$P\left(\left|\frac{1}{N\beta} \ln Z_N - \frac{1}{N\beta} E \ln Z_N\right| \geqslant u\right) \leqslant 2e^{N(u-\alpha(1+\frac{u}{\alpha})\ln(1+\frac{u}{\alpha}))}. \tag{100}$$

Remarks. The theorem can be immediately extended to the more general class of diluted spin glass models considered in ref. 3. In particular, for the diluted p-spin model[23] with $p \geqslant 3$ the above result holds without any modification, while for the K-sat model one has to replace (100) by

$$P\left(\frac{\ln Z_N}{N\beta}-\frac{E\ln Z_N}{N\beta}\leqslant -u\right)\leqslant e^{N(u-\alpha(1+\frac{u}{\alpha})\ln(1+\frac{u}{\alpha}))} \qquad u>0 \tag{101}$$

$$P\left(\frac{\ln Z_N}{N\beta}-\frac{E\ln Z_N}{N\beta}\geqslant u\right)\leqslant e^{N(-u-\alpha(1-\frac{u}{\alpha})\ln(1-\frac{u}{\alpha}))} \qquad 0<u<\alpha \tag{102}$$

$$P\left(\frac{\ln Z_N}{N\beta}-\frac{E\ln Z_N}{N\beta}\geqslant u\right)=0 \qquad u\geqslant \alpha. \tag{103}$$

(The latter is a simple consequence of the fact that, for the K-sat, $1/N\ln Z_N\leqslant \ln 2$ for any disorder realization, and that $1/NE\ln Z_N\geqslant \ln 2-\alpha\beta$, as it is easily verified from the definition of the model.) In particular, Eqs. (101)–(102) allow to recover the bound given in ref. 24 for the fluctuations of the minimal fraction of unsatisfied clauses in the K-sat problem.

Proof of Theorem 5. We sketch just the main steps in the proof, since it is very similar in spirit to that given for fully connected models in refs. 14 and 25, the main difference being that the role of Gaussian integration by parts is replaced here by the properties (13), (14) of Poisson random variables.

Introduce the interpolating parameter $0\leqslant t\leqslant 1$ and define, for $s\in\mathbb{R}$,

$$\varphi_N(t)=\ln E_1\exp\{sE_2\ln Z_N(t)\}, \tag{104}$$

where

$$Z_N(t)=\sum_{\{\sigma\}}\exp\beta\left(\sum_{\mu=1}^{\xi^1_{2\alpha Nt}}J^1_\mu\sigma_{i^1_\mu}\sigma_{j^1_\mu}+\sum_{\nu=1}^{\xi^2_{2\alpha N(1-t)}}J^2_\nu\sigma_{i^2_\nu}\sigma_{j^2_\nu}\right). \tag{105}$$

Here, all variables with upper index 1 are independent from those with index 2, and E_ℓ denotes the average

$$E_\ell(.)=E_{\xi^\ell}E_{\{J^\ell_\mu\}}E_{\{i^\ell_\mu\}}E_{\{j^\ell_\mu\}}(.), \qquad \ell=1,2.$$

The motivation for the introduction of $\varphi_N(t)$ is the identity

$$\exp\{\varphi_N(1)-\varphi_N(0)\}=E\exp\{s(\ln Z_N-E\ln Z_N)\}. \tag{106}$$

Since we want to bound from above the r.h.s. of (106), we compute the t derivative of $\varphi_N(t)$. After some straightforward computations, one finds

$$\varphi'_N(t) = \alpha \frac{\sum_{i,j=1}^N}{N} \frac{E_1\{e^{sE_2 \ln Z_N(t)} E_J(e^{sE_2 \ln \omega(e^{\beta J \sigma_i \sigma_j})} - 1 - sE_2 \ln \omega(e^{\beta J \sigma_i \sigma_j}))\}}{E_1 \exp\{sE_2 \ln Z_N(t)\}}$$

and, thanks to the trivial bounds

$$-\beta \leqslant E_2 \ln \omega(e^{\beta J \sigma_i \sigma_j}) \leqslant \beta$$

one has

$$|\varphi'_N(t)| \leqslant \alpha N(e^{|s| \beta} - 1 - |s| \beta). \tag{107}$$

Putting together Eqs. (107) and (106), employing Tchebyshev's inequality and optimizing on s, one finally finds the statement of the theorem. ∎

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REFERENCES

1. M. Mézard, G. Parisi, and M. A. Virasoro, *Spin Glass Theory and Beyond* (World Scientific, Singapore, 1987).
2. M. Mézard, G. Parisi, and R. Zecchina, Analytic and algorithmic solution of random satisfiability problems, *Science* **297**:812–815 (2002).
3. S. Franz and M. Leone, Replica bounds for optimization problems and diluted spin systems, *J. Stat. Phys.* **111**:535 (2003).
4. M. Talagrand, The high temperature case of the K-sat problem, *Probab. Theory Related Fields* **119**:187–212 (2001).
5. F. Guerra and F. L. Toninelli, The thermodynamic limit in mean field spin glass models, *Comm. Math. Phys.* **230**:71–79 (2002).
6. F. Guerra, Broken replica symmetry bounds in the mean field spin glass model, *Comm. Math. Phys.* **233**:1–12 (2003).
7. L. Viana and A. J. Bray, Phase diagrams for dilute spin-glasses, *J. Phys. C* **18**:3037–3051 (1985).
8. I. Kanter and H. Sompolinsky, Mean-field theory of spin-glasses with finite coordination number, *Phys. Rev. Lett.* **58**:164 (1987).
9. D. Sherrington and S. Kirkpatrick, Solvable model of a spin-glass, *Phys. Rev. Lett.* **35**: 1792–1796 (1975).

10. F. Guerra and F. L. Toninelli, Quadratic replica coupling for the Sherrington–Kirkpatrick mean field spin glass model, *J. Math. Phys.* **43**:3704–3716 (2002).
11. F. Guerra and F. L. Toninelli, Central limit theorem for fluctuations in the high temperature region of the Sherrington–Kirkpatrick spin glass model, *J. Math. Phys.* **43**:6224–6237 (2002).
12. N. Creignou and H. Daude, Satisfiability threshold for random XOR-CNF formulas, *Discr. Appl. Math.* **96–97**:41–53 (1999).
13. G. Semerjian and L. F. Cugliandolo, Cluster expansions in dilute systems: applications to satisfiability problems and spin glasses, *Phys. Rev. E* **64**:036115 (2001).
14. M. Talagrand, Mean field models for spin glasses: A first course, to appear in *Proceedings of the 2000 Saint Flour Summer School in Probability*.
15. F. Comets and J. Neveu, The Sherrington–Kirkpatrick model of spin glasses and stochastic calculus: The high temperature case, *Comm. Math. Phys.* **166**:549–564 (1995).
16. M. Talagrand, *Spin Glasses: A Challenge for Mathematicians. Cavity and Mean Field Models* (Springer-Verlag, 2003).
17. F. Guerra, The cavity method in the mean field spin glass model. Functional representation of the thermodynamic variables, in *Advances in Dynamical Systems and Quantum Physics*, S. Albeverio *et al.*, eds. (World Scientific, Singapore, 1995).
18. M. Talagrand, Exponential inequalities and replica symmetry breaking for the Sherrington–Kirkpatrick Model, *Ann. Probab.* **28**:1018–1062 (2000).
19. M. Aizenman, J. Lebowitz, and D. Ruelle, Some rigorous results on the Sherrington–Kirkpatrick spin glass model, *Comm. Math. Phys.* **112**:3–20 (1987).
20. R. Monasson and R. Zecchina, Statistical mechanics of the random K-satisfiability model, *Phys. Rev. E* **56**:1357 (1997).
21. M. Talagrand, Concentration of measure and isoperimetric inequalities in product spaces, *Publ. Math. I.H.E.S.* **81**:73–205 (1995).
22. W. Rhee and M. Talagrand, Martingale inequalities and NP-complete problems, *Math. Oper. Res.* **12**:177–181 (1987).
23. F. Ricci-Tersenghi, M. Weigt, and R. Zecchina, Simplest random K-satisfiability problem, *Phys. Rev. E* **63**:026702 (2001).
24. A. Z. Broder, A. M. Frieze, and E. Upfal, On the satisfiability and maximum satisfiability of random 3-CNF formulas, in *Proceedings of the 4th Annual ACM-SIAM Symposium on Discrete Algorithms* (Association for Computing Machinery, New York, 1993), pp. 322–330.
25. F. Guerra and F. L. Toninelli, The infinite volume limit in generalized mean field disordered models, *Markov Process. Related Fields* **9**:195–207 (2003).